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Back-Reaction Equations for Isotropic Cosmologies when Nonconformal Particles Are Created

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A model proposed some years ago by Hartle to study the back reaction in a cosmological model due to the creation of massless non-conformally coupled particles is reexamined. The model consists of a spatially flat FRW spacetime with a classical source made of two perfect fluids one a radiative fluid and the other a baryonic fluid with the equation of state of dust, and it is assumed that the ratio of baryons to photons is small. The back-reaction equations for the cosmological scale factor are derived using a CTP (closed time path) effective action method. Making use of the connection, in the semiclassical context, between the CTP effective action and the influence functional in quantum statistical mechanics, improved back-reaction equations are derived which take into account the fluctuations of the stress-energy tensor of the quantum field. These new dynamical equations are real and causal and predict stochastic fluctuations for the cosmological scale factor.

1. INTRODUCTION

The back reaction of quantum fields on the dynamics of the very early universe could be very important. The study of quantum effects such as the creation of particles due to the universe expansion has a long history that goes back to the early work by Parker (1968, 1969) for homogeneous and isotropic FRW (Friedmann–Robertson–Walker) cosmologies, and to Zeldovich and Starobinsky (1971, 1977) (see also Birrell and Davies, 1980) for homogeneous but anisotropic cosmologies. It was first argued by Zeldovich (1970) that the back reaction due to the creation of particles in anisotropic

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models is a possible mechanism for the dissipation of the anisotropies in the early universe, and in fact it was soon shown that this could be the case (Lukash and Starobinsky, 1974; Lukash *et al.*, 1976; Hu and Parker, 1978), provided one extrapolated the calculations to very early times.

A systematic approach to the back-reaction problem in cosmology was initiated by Hartle (1977), who proposed to use the effective action to one-loop order to derive the semiclassical Einstein equations that describe the back reaction of the quantum fields on the dynamics of the gravitational field. This suggestion was followed by a study of the back-reaction problem for different cosmological scenarios such as conformally coupled fields in FRW (Fischetti *et al.*, 1979) or conformally invariant fields in cosmologies with small anisotropies (Hartle and Hu, 1979, 1980a, b). As expected, in the models where conformal invariance was broken by the anisotropies particles were created and a rapid dissipation of these anisotropies took place very early in the universe.

However, since the source of gravity in the semiclassical theory is the renormalized expectation value in some quantum state of the stress-energy momentum tensor of the quantum field, and this tensor contains terms quadratic in the spacetime curvature (as a consequence of the renormalization), the back-reaction effects on the classical geometry may be important even when there is no production of particles. We should recall that there is a two-parameter ambiguity in the expectation value of such a stress-energy tensor that cannot be resolved within the semiclassical theory (Birrell and Davies, 1982; Wald, 1994); these parameters are related to two local tensors which are conserved and are quadratic in the curvature. In FRW cosmologies once these parameters are fixed, the equations for the cosmological scale factor admit a two-parameter family of solutions.

Thus, Starobinsky (1980) showed that inflationary de Sitter models (Starobinsky inflation) which evolved into classical matter-dominated FRW cosmologies would be the consequence of the back reaction due to purely conformally coupled quantum fields. Detailed numerical calculations by Anderson (1983, 1984) showed that assuming FRW spacetimes which evolved into the future as classical solutions driven by classical radiation could have very different behaviors from the classical solution at early times just assuming the back reaction of conformally coupled quantum fields. Some solutions start at an initial singularity and have no particle horizons, others avoid the singularity, whereas some start as contracting de Sitter universes and bounce once, while others bounce an infinite number of times and have initial singularities, but no particle horizons. In models where massive conformally coupled fields are considered, the calculations are more involved because there are nonlocal effects associated to particle creation, but the results are qualitatively similar (Anderson, 1985, 1986). The validity, however, of solutions to the back-reaction equations that deviate nonperturbatively from the classical solutions has been questioned in recent years (Simon, 1991; Parker and Simon, 1993; Flanagan and Wald, 1996).

An interesting model was proposed in 1981 by Hartle to study the back reaction of massless non-conformally coupled quantum fields in cosmology. The model is simple enough so that some results may be derived analytically and yet it has some of the features of a realistic cosmology. The model consists of a spatially flat FRW spacetime with a classical source made of two perfect fluids, one of which has the equation of state of radiation and the other represents baryonic matter with the equation of state of dust. A dimensionless parameter $\hat{\xi}$ measuring the relative amounts of baryons and radiation, which is constant in classical periods, is assumed to be always constant and with a value corresponding to the present universe which is very small. A massless nonconformal field is coupled to this system. The presence of the small portion of baryonic matter is essential to ensure particle creation (a radiative FRW universe has a vanishing scalar curvature and the fields do not couple to the curvature). A perturbative expansion in terms of the parameter $\hat{\xi}$ is then seen to be equivalent to an expansion in a parameter that measures the deviation from the conformal coupling.

Since the two degrees of freedom of the graviton field in an FRW background behave as massless minimally coupled fields (Grishchuk, 1974), Hartle's model provides a good testing ground for the study of the back reaction due to the creation of gravitons in cosmological models. In this model, of course, there is only one dynamical variable, the cosmological scale factor, which depends on time only. From the equations for this scale factor Hartle showed that solutions existed that evolved in time toward the classical FRW expansion for a universe with matter and radiation. However, at early times even when imposing that the departure from the classical solution is small (i.e., considering only the perturbative solutions), quantum effects near the initial singularity are important enough to soften the classical singularity in such a way that the rate of particle production is finite, whereas such a rate would be infinite if the back reaction of the quantum fields were ignored.

Here we reconsider Hartle's model using a closed time path (CTP) effective action method to derive the dynamical equation for the scale factor. The advantage of this method over the "in–out" effective action method used by Hartle (1981) is that the dynamical equations are now real and causal. This makes the interpretation of the dynamical equations and their solutions much simpler and such equations can now be formulated as an initial value problem. The CTP effective action was first discussed by Schwinger (1961, 1962) and Keldysh (1964). For an application of the CTP effective action method to the back-reaction problem in different cosmological models see

Jordan (1986, 1987), Calzetta and Hu (1987, 1989), Paz (1990), and Campos and Verdaguer (1994).

The CTP formalism allows also to derive in a very natural way the stochastic effects that the quantum field induce on the cosmological scale factor. In fact, classical stochastic fluctuations in the gravitational field are predicted by the semiclassical Einstein–Langevin equations. These equations have been recently derived in different cosmological contexts as modifications to the semiclassical equations to account, to some extent, for the quantum fluctuations of the expectation value of the stress-energy tensor (Calzetta and Hu, 1995; Hu and Matacz, 1995; Hu and Sinha, 1995; Campos and Verdaguer, 1996). Since these fluctuations are of quantum origin, we expect that they will be significant at the very early universe.

An important step to derive the Langevin type of equations in the semiclassical back-reaction problem was made by Hu (1989), who proposed to view this problem in the light of quantum open systems. In fact, in the semiclassical context we have the interaction of two systems, one of which is the gravitational field, and the other is the quantum field. Since only the dynamics of the first is of interest (the "system"), we can integrate out the degrees of freedom of the other system (the "environment"). All we need to know is the influence of the "environment" on the "system." A functional formalism developed by Feynman and Vernon (1963; Feynmann and Hibbs, 1965), which is called the influence functional formalism, is already at hand for this. The influence functional formalism is closely related to the CTP functional formalism and it can be shown that the imaginary part of the CTP effective action contains the terms responsible for the stochastic fluctuations of the gravitational field. In all cases studied one has been able to prove a fluctuation-dissipation relation which connects the dissipation suffered by the gravitational field as a consequence of the production of quantum particles with the reaction by means of a stochastic source that the quantum field produces on the gravitational field. This is very much like the quantum Brownian motion of a particle moving in a bath of quantum oscillators (Caldeira and Leggett, 1983). The influence functional method has been frequently used in recent years to derive Langevin-type equations in connection with the study of out-of-equilibrium quantum systems in the early universe (Morikawa, 1986; Lee and Boyanovsky, 1993; Gleiser and Ramos, 1994; Boyanovsky *et al.*, 1995).

The plan and a short summary of the paper are the following. In Section 2 we describe Hartle's model and give the relevant terms for the action, which involves the quantum scalar field, the gravitational field, and the classical matter sources. In Section 3 we briefly summarize the calculations leading to the CTP effective action for the cosmological scale factor and give the final expression for such an effective action. In Section 4 we use the previous results to give a fluctuation-dissipation relation in this case. We also use the relation between the CTP effective action and the influence functional of Feynman and Vernon to derive an improved effective action for the cosmological scale factor which takes into account the fluctuations of the stress-energy tensor of the quantum field by including a coupling of the scale factor to a stochastic source. Finally, in Section 5 we derive the stochastic back-reaction equations from this improved effective action and compare our results with those by Hartle. We see that our equations are real and causal and that when the equations are averaged with respect to the stochastic source, they lead to the same behavior for small and large times as Hartle's equations. Now, however, there is a new stochastic term which predicts stochastic fluctuations for the cosmological scale factor; the detailed solutions in this case and the consequences of such stochastic behavior will be the subject of further research.

2. THE MODEL

In this section we describe Hartle's (1981) model in which a massless nonconformal quantum scalar field is coupled to a spatially flat FRW cosmology with a classical source made of radiation and dust. Since we are interested in the effective action for the cosmological scale factor due to the quantum field, we need to add counterterms to the gravitational action to renormalize the effective action. We use a dimensional regularization technique and thus we write the relevant terms in the classical action in n arbitrary dimensions. The cosmological model is described by the n-dimensional spatially flat FRW metric

$$ds^2 = a^2(\eta)[-d\eta^2 + d\mathbf{x}^2] \tag{2.1}$$

where $a(\eta)$ is the cosmological scale factor, and η is the conformal time, which is related to cosmological time t by $a\, d\eta = dt$. The classical action for the massless scalar matter field $\Phi(x^\mu)$ non-conformally coupled is

$$S_m[g_{\nu\mu}, \Phi] = -\frac{1}{2}\int d^n x\sqrt{-g}\left[g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + \left(\frac{n-2}{4(n-1)} + \nu\right)R\Phi^2\right] \tag{2.2}$$

where R is the Ricci curvature scalar, and $(n-2)/4(n-1) + \nu$ is the parameter coupling of the scalar field to the spacetime curvature; $\nu = 0$ corresponds to a conformally coupled field in n dimensions. Since the metric (2.1) is conformally flat, it is convenient to introduce a new rescaled matter field $\phi(x^\mu)$ by

$$\phi(x^\mu) \equiv a^{(n-2)/2}(\eta)\Phi(x^\mu) \tag{2.3}$$

and the scalar field action simplifies considerably,

$$S_m[a, \phi] = \frac{1}{2}\int d^n x\, \phi(x^\mu)[\Box - \nu a^2 R]\phi(x^\mu) \tag{2.4}$$

where now $\Box \equiv \eta_{\mu\nu}\partial_\mu\partial_\nu$ is the flat d'Alambertian operator. It is clear from this equation that if $\nu \neq 0$, the conformally flat symmetry is broken by the coupling of the scalar field.

As we have mentioned, the action for the gravitational field, the Einstein–Hilbert action, needs to be corrected with a counterterm to cancel the divergences that will come from the effective action. In our case it suffices to add a term quadratic in the Ricci scalar. Note that terms quadratic in $R_{\mu\nu}$ or the Weyl tensor could also be added, but since these terms are not necessary for the renormalization, we simply assume that their coefficients vanish. This is related to the two-parameter ambiguity in the renormalized expectation value of the stress-energy tensor of the quantum field (Birrell and Davies, 1982; Wald, 1994) that we have already mentioned; nevertheless, we will see that part of such ambiguity remains here. Thus the Einstein–Hilbert action plus the divergent (in $n = 4$) counterterms are

$$S_g[a; \mu_c] = \frac{1}{l_p^2}\int d^n x\sqrt{-g}R + \frac{\nu^2\mu_c^{n-4}}{32\pi^2(n-4)}\int d^n x\sqrt{-g}R^2 \tag{2.5}$$

where $l_p = (16\pi G)^{1/2}$ is the Planck length and μ_c is an arbitrary mass scale needed to give the correct dimensions to the counterterm. Since the only gravitational variable is the cosmological scale factor $a(\eta)$, the above equation reduces in our case, when expanded in $(n - 4)$, to

$$S_g^{\mathrm{EH}}[a] = \frac{6\mathcal{V}}{l_p^2}\int d\eta\, a\ddot{a} \tag{2.6}$$

$$\begin{aligned} S_g^{\mathrm{div}}[a; \mu_c] = \frac{\nu^2\mathcal{V}}{32\pi^2}\Bigg[&\frac{36}{n-4}\int d\eta\left(\frac{\ddot{a}}{a}\right)^2 + 36\ln\mu_c\int d\eta\left(\frac{\ddot{a}}{a}\right)^2 \\ &+ 12\int d\eta\left\{3\ln a\left(\frac{\ddot{a}}{a}\right)^2 + \left(\frac{\ddot{a}}{a}\right)\left[3\left(\frac{\dot{a}}{a}\right)^2 + 2\left(\frac{\ddot{a}}{a}\right)\right]\right\}\Bigg] \\ &+ O(n-4) \end{aligned} \tag{2.7}$$

where $\mathcal{V}$ is the volume integral $\mathcal{V} \equiv \int d^3\mathbf{x}$ and we have separated the Einstein–Hilbert action from the divergent term.

To the above scalar and gravitational actions we need to add the action for the classical matter sources $S_m^{\rm cl}$, which are radiative and dust perfect fluids. This action is given by

$$S_m^{\rm cl}[a] = -\mathcal{V}\int d\eta\ \tilde{\rho}_b a \tag{2.8}$$

where $\tilde{\rho}_b$ is a constant parameter related to the baryon energy density. It is connected to a similar parameter $\tilde{\rho}_r$ for the energy density of the radiation through

$$\hat{\xi} = \frac{l_p\tilde{\rho}_b}{\tilde{\rho}_r^{3/4}} \tag{2.9}$$

which is of order of $\hat{\xi} \sim 10^{-27}$ for the present universe. This parameter $\hat{\xi}$ measures the ratio of baryons to photons.

The action (2.8) is justified because it reproduces the trace of the stress-energy tensor for a radiative perfect fluid with equation of state $p_r = \rho_r/3$, where p_r is the pressure and ρ_r is the energy density of the radiation, and of dust-like baryon fluid of pressure ρ_b and energy density p_b with an equation of state $p_b = 0$, when varied with respect to the scale factor $a(\eta)$. When the dynamics of the scale factor is driven by the classical source only, one finds that $\rho_b = \tilde{\rho}_b a^{-3}$ and $\rho_r = \tilde{\rho}_r a^{-4}$ and then the relative amounts of baryons and radiation are defined by $l_p\rho_b/\rho_r^{3/4}$ becoming constant and given by (2.9). In fact, when the classical action (2.8) is considered as the only dynamical source of gravity from the Einstein–Hilbert action (2.6) one gets $6\ddot{a} = (l_p^2/2)\tilde{\rho}_b$ or equivalently $R = -(l_p^2/2)T^{\rm cl}$, where $T^{\rm cl} = -\rho_b$ is the trace of the stress-energy tensor of a perfect fluid of dust (baryons) and radiation.

We note that when baryons are not present, i.e., when $\hat{\xi} = 0$, the scalar curvature ($R = 0$), and from the action (2.4) for the scalar field $\phi(x^\mu)$ it is clear that this field behaves as a free field in flat spacetime. Therefore with the usual Minkowskian definition of a vacuum state [which corresponds to the conformal vacuum for $\Phi(x^\mu)$] no particles can be created by the universe expansion. Back-reaction effects are still possible due to the vacuum expectation value of the stress-energy tensor of the scalar field, which in this case is formed by terms purely quadratic in the curvature and which depend on the two parameters of the above-mentioned parameter ambiguity. One of the parameters is associated with an action proportional to R^2 and the other with an action proportional to $R_{\mu\nu}R^{\mu\nu}$. The first vanishes when $R = 0$, but the second is generally different from zero even when the scalar curvature vanishes. Thus all quantum effects in this case depend crucially on the second parameter being different from zero. We will assume that that such parameter is zero and then $R = 0$ is a consistent solution to the semiclassical back-reaction problem. Since our interest is in the back reaction due to particle production, we will take $\hat{\xi}$ as a small nonzero parameter, and we will consider the $\nu a^2 R$ term in (2.4) as a perturbative term which gives a measure of the deviation from the radiative case $R = 0$. In practice we will compute the quantum corrections as perturbations in powers of the parameter ν, and this should be consistent with an expansion in $\hat{\xi}$.

3. CTP EFFECTIVE ACTION

In this section we compute the closed time path (CTP) effective action to one-loop order for our massless quantum field $\Phi(x^\mu)$ using a perturbative expansion in powers of the parameter ν. The CTP effective action was introduced by Schwinger (1961, 1962; Keldysh, 1964; Chou *et al.*, 1985). For its application in a curved background see Jordan (1986, 1987), Calzetta and Hu (1987, 1989), Paz (1990), and Campos and Verdaguer (1994), and for the details in this paper we closely follow Campos and Verdaguer (1994).

The idea is to start with a generating functional from which one may obtain expectation values, instead of matrix elements, as one obtains using the generating functional of the ordinary in–out effective action technique. For this one lets a certain in-vacuum evolve independently under two different external classical sources J_+ and J_-. In terms of a certain general field $\psi(x)$ one can give a path integral representation of such a generating functional as

$$e^{iW[J_+,J_-]} = \int \mathcal{D}[\psi_+]\,\mathcal{D}[\psi_-]e^{i(S[\psi_+]-S[\psi_-]+J_+\psi_+-J_-\psi_-)} \tag{3.1}$$

where $S[\psi]$ is the action for the field ψ, $J_\pm\psi_\pm \equiv \int d^nx\, J_\pm(x)\psi_\pm(x)$, and it is understood that we sum over all fields ψ_+ and ψ_- with negative and positive frequency modes, respectively, in the remote past, but which coincide at some future time (usually at $t \to +\infty$). It is easy to see that expectation values are obtained from this generating functional. This formalism is called CTP because the above path integral can be seen as the path sum of one field evolving in two different time branches: one going forward in time in the presence of J_+ and then backward in time in the presence of J_- from the future to the in-vacuum.

The CTP effective action is the Legendre transform of this functional

$$\Gamma_{\mathrm{CTP}}[\bar{\psi}_+, \bar{\psi}_-] = W[J_+, J_-] - J_+\bar{\psi}_+ + J_-\bar{\psi}_- \tag{3.2}$$

where the field $\bar{\psi}$ is defined by $\bar{\psi}_\pm \equiv \pm(\delta W/\delta J_\pm)$. The dynamical equation for the vacuum expectation value of the field $\bar{\psi}[0] \equiv \bar{\psi}_\pm[0] = \langle 0, \mathrm{in}|\psi(x)|0, \mathrm{in}\rangle$ is simply obtained as

$$\left.\frac{\delta\Gamma_{\mathrm{CTP}}[\bar{\psi}_+, \bar{\psi}_-]}{\delta\bar{\psi}_+}\right|_{\bar{\psi}_\pm=\bar{\psi}[0]} = 0 \tag{3.3}$$

In our case we have two fields $a(\eta)$ and $\phi(x^\mu)$ whose free actions are given by $S_g[a; \mu_c] + S_m^{cl}[a]$ from (2.6)–(2.8), and by $S_m^{free}[\phi] = 1/2 \int d^nx\, \phi\Box\phi$ from (2.4), and whose interaction action is given by $S_{int}[a, \phi] = -\nu/2 \int d^nx\, a^2R\phi^2$. Thus the generic field ψ above must be seen now as two fields, but since we work in the semiclassical approximation we will only quantize the scalar field ϕ, whereas a will be considered as a classical field.

Thus we can proceed to the evaluation of the effective action up to one-loop order for the field ϕ, which corresponds to the first-order expansion of the generating functional in powers of $\hbar$. As usual (Abers and Lee, 1973) we take $\phi_\pm^{(0)}$ as the classical solutions and expand the exponent in (3.1) about these background fields up to the second derivative of $S_m[\phi_\pm]$ with respect to $\phi_\pm$; then the integration in $\phi_\pm$ is Gaussian. Let $G(x, y)$ be the propagator, i.e., the inverse of the classical kinetic operator $A = \mathrm{diag}(\Box - \nu a^{+2}R^+, -(\Box - \nu a^{-2}R^-))$; see (2.4). We should note that since we have two fields ϕ_+ and ϕ_-, G is a matrix operator with 2×2 components. Thus we get $W[J_\pm] \simeq W^{(o)}[J_\pm] - (i/2)\mathrm{Tr}(\ln G)$, where $W^{(o)}$ represents the classical action with external sources, and from this, by explicit evaluation of the Legendre transform to the same order, we obtain the CTP effective action as

$$\Gamma_{CTP}[a^\pm, \overline{\phi}_\pm] \simeq S_g[a^+; \mu_c] - S_g[a^-; \mu_c] + S_m^{cl}[a^+] - S_m^{cl}[a^-] + S_m[a^+, \overline{\phi}_+] - S_m[a^-, \overline{\phi}_-] - \frac{i}{2}\mathrm{Tr}(\ln G) \qquad (3.4)$$

Since the action (2.4) is quadratic in the field, the one-loop-order result (3.4) is exact in this case. However, the propagator G cannot be found exactly, because of the interaction term in (2.4), and thus we expand G in powers of ν. Let us define

$$V(a) \equiv -\nu a^2 R \qquad (3.5)$$

Then up to second order in ν, $G = G^o[1 - VG^o + VG^oVG^o + \cdots]$, where G^o is the flat spacetime propagator $(G^o)^{-1} = \mathrm{diag}(\Box, -\Box)$; its four components are $G^o_{++} = \Delta_F$, $G^o_{--} = -\Delta_D$, $G^o_{+-} = -\Delta^+$, and $G^o_{-+} = \Delta^-$ where Δ_F and Δ_D are the Feynman and Dyson propagators, respectively, and $\Delta^\pm$ are the Wightman functions. Substituting the above expansion in (3.4), we have up to second order in ν

$$\begin{aligned}\Gamma_{CTP}[a^\pm, \overline{\phi}_\pm] \simeq{}& S_g[a^+; \mu_c] - S_g[a^-; \mu_c] + S_m^{cl}[a^+] - S_m^{cl}[a^-] \\ &+ S_m[a^+, \overline{\phi}_+] - S_m[a^-, \overline{\phi}_-] \\ &- \frac{i}{2}\mathrm{Tr}(\ln G^o_{ab}) + \frac{i}{2}\mathrm{Tr}(V_+G^o_{++}) - \frac{i}{2}\mathrm{Tr}(V_-G^o_{--})\end{aligned}$$

$$- \frac{i}{4}\,\mathrm{Tr}(V_+ G^o_{++} V_+ G^o_{++}) - \frac{i}{4}\,\mathrm{Tr}(V_- G^o_{--} V_- G^o_{--})$$

$$+ \frac{i}{2}\,\mathrm{Tr}(V_+ G^o_{+-} V_- G^o_{-+}) \tag{3.6}$$

This effective action depends on the fields $\overline{\phi}_\pm$ and the scale factor $a^\pm$. The dynamical equation for the field $\overline{\phi}(x)$ may be obtained by functional derivation with respect to $\overline{\phi}_+$ as explained in (3.3), and it is clear from (3.6) that the vacuum expectation value $\overline{\phi} \equiv \langle 0, \mathrm{in}|\phi|0, \mathrm{in}\rangle = 0$. Since our main interest is the dynamical equation for $a(\eta)$, we can substitute the above expression in the effective action and we obtain an effective action which is a functional of $a^\pm$ only, i.e., $\Gamma_{\mathrm{CTP}}[a^\pm]$ of (3.6). As explained in more detail in Campos and Verdaguer (1994), we only need to evaluate the last three terms because the rest do not contribute to the dynamical equations for $a(\eta)$. These terms are

$$\begin{aligned} T^\pm &= -\frac{i}{4}\,\mathrm{Tr}(V_\pm G^o_{\pm\pm} V_\pm G^o_{\pm\pm}) \\ &= \mp\frac{\nu^2}{32\pi^2}\left\{\frac{36\mathcal{V}}{n-4}\int d\eta\left(\frac{\ddot{a}^\pm}{a^\pm}\right)^2 + 12\mathcal{V}\int d\eta\left(\frac{\ddot{a}^\pm}{a^\pm}\right)\left[3\left(\frac{\dot{a}^\pm}{a^\pm}\right)^2 + 2\left(\frac{\ddot{a}^\pm}{a^\pm}\right)\right]\right. \\ &\quad \left. - 36\int d\eta\, d\eta'\left(\frac{\ddot{a}^\pm}{a^\pm}\right)(\eta)\mathrm{K}^\pm(\eta-\eta';\mu_o)\left(\frac{\ddot{a}^\pm}{a^\pm}\right)(\eta') + O(n-4)\right\} \end{aligned} \tag{3.7}$$

$$\begin{aligned} T &= \frac{i}{2}\,\mathrm{Tr}(V_+ G^o_{+-} V_- G^o_{-+}) \\ &= \frac{9\nu^2}{4\pi^2}\int d\eta\, d\eta'\left(\frac{\ddot{a}^+}{a^+}\right)(\eta)\mathrm{K}(\eta-\eta')\left(\frac{\ddot{a}^-}{a^-}\right)(\eta') + O(n-4) \end{aligned} \tag{3.8}$$

where $\mu_o^2 = \exp(2 + \ln 4\pi - \gamma)$ and

$$\begin{aligned} K^\pm(\eta-\eta';\mu_o) &= 16\pi^2[\mathrm{A}(\eta-\eta';\mu_o) \pm i\mathrm{N}(\eta-\eta')] \\ K(\eta-\eta') &= -16\pi^2[\mathrm{D}(\eta-\eta') + i\mathrm{N}(\eta-\eta')] \end{aligned} \tag{3.9}$$

with

$$\begin{aligned} \mathrm{A}(\eta-\eta';\mu_o) &= -\frac{\mathcal{V}}{32\pi^2}\int_{-\infty}^{\infty}\frac{dp^o}{2\pi}\,e^{-ip^o\cdot(\eta-\eta')}\ln\left(\frac{p^o}{\mu_o}\right)^2 \\ \mathrm{D}(\eta-\eta') &= \frac{i\mathcal{V}}{32\pi}\int_{-\infty}^{\infty}\frac{dp^o}{2\pi}\,e^{-ip^o\cdot(\eta-\eta')}\,\mathrm{sgn}(-p^o) \end{aligned}$$

$$\mathrm{N}(\eta - \eta') = \frac{\mathcal{V}}{32\pi}\,\delta(\eta - \eta') \tag{3.10}$$

The two kernels $\mathrm{K}^{\pm}(\eta - \eta'; \mu_o)$ come from the evaluation of $[\Delta_{F/D}(x - x')]^2$, and $\mathrm{K}(\eta - \eta')$ comes from the evaluation of $[\Delta^{+}(x - x')]^2$, which appear in (3.7) and (3.8), respectively, followed by space integrations.

The divergent terms when $n \to 4$ of (3.7) are canceled by the terms quadratic in the curvature introduced in the gravitational part of the action $S_g[a^{\pm}; \mu_c]$. Finally, the renormalized CTP effective action is

$$\Gamma_{\mathrm{CTP}}[a^{\pm}] = S^{\mathrm{R}}_{g,m}[a^{+}] - S^{\mathrm{R}}_{g,m}[a^{-}] + S^{\mathrm{R}}_{\mathrm{IF}}[a^{\pm}] \tag{3.11}$$

where

$$S^{\mathrm{R}}_{g,m}[a] = \mathcal{V} \int d\eta \left[-\frac{6}{l_p^2}\,\dot{a}^2 + \frac{9\nu^2}{8\pi^2} \ln a \left(\frac{\ddot{a}}{a}\right)^2 - \tilde{\rho}_b a \right] \tag{3.12}$$

and

$$S^{\mathrm{R}}_{\mathrm{IF}}[a^{\pm}] = \frac{1}{32\pi^2} \int d\eta\, d\eta'\, \Delta V(\eta) \mathrm{H}(\eta - \eta'; \bar{\mu}) \{V(\eta')\}$$
$$+ i \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{n} d\eta'\, \Delta V(\eta) \mathrm{N}(\eta - \eta') \Delta V(\eta') \tag{3.13}$$

where $\bar{\mu} = \mu_o \mu_c$, we have used the notation $\Delta V(\eta) = V_{+}(\eta) - V_{-}(\eta)$, $\{V(\eta)\} = V_{+}(\eta) + V_{-}(\eta)$, with $V(\eta) = 6\nu(\ddot{a}/a)$, and the new kernel $\mathrm{H}(\eta - \eta'; \bar{\mu})$ is

$$\mathrm{H}(\eta - \eta'; \bar{\mu}) = 16\pi^2[A(\eta - \eta'; \bar{\mu}) - \mathrm{D}(\eta - \eta')] \tag{3.14}$$

The subscript IF in $S^{\mathrm{R}}_{\mathrm{IF}}[a^{\pm}]$ is used to indicate that this part of the CTP action is the influence functional action as defined by Feynman and Vernon (1963). That is, it gives the effect of the "environment," the quantum field, on the "system," the cosmological scale factor, which is the field of interest here. The identification between a part of the CTP effective action and the influence functional was proved by Calzetta and Hu (1994).

4. IMPROVED EFFECTIVE ACTION

The connection between the CTP effective action in the semiclassical context and the influence functional action introduced by Feynman and Vernon to describe the interaction between a "system" and an "environment" just noted gives an interesting new light on the semiclassical back-reaction problem, as first noted by Hu (1989).

4.1. Fluctuation-Dissipation Relation

In analogy with the quantum Brownian model (Hu *et al.*, 1993), a fluctuation-dissipation relation can be derived in this case. In fact, we can identify the noise kernel as the kernel $N(\eta - \eta')$ of the imaginary part of the influence functional action (3.13), $\mathrm{Im}(S^{R}_{IF})$, which is quadratic in the field difference $\Delta V = V_{+} - V_{-}$. To identify the dissipation kernel, it is convenient to write the real part of the influence functional as follows:

$$\mathrm{Re}(S^{R}_{IF}[a^{\pm}]) = \frac{1}{32\pi^2}\int d\eta\, d\eta'\, [V_{+}(\eta)V_{+}(\eta') - V_{-}(\eta)V_{-}(\eta')]\hat{\gamma}(\eta - \eta'; \bar{\mu}) - \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\eta} d\eta'\, \Delta V(\eta) D(\eta - \eta')\{V(\eta')\} \tag{4.1}$$

where

$$\hat{\gamma}(\eta - \eta'; \bar{\mu}) = 16\pi^2\,[A(\eta - \eta'; \bar{\mu}) + D(\eta - \eta')\,\mathrm{sgn}(\eta - \eta')]$$

i.e., we have separated the kernel $H(\eta - \eta'; \bar{\mu})$ into its symmetric and antisymmetric parts in the interchange of η by η'.

The second term in (4.1) is written now in the standard form (Feynman and Vernon, 1963; Feynman and Hibbs, 1965) to identify $D(\eta - \eta')$ as the dissipation kernel. This term is not symmetric under the change of temporal order. The explicit fluctuation-dissipation relation may be derived as follows. First we note that $D(\eta)$ may be written as a time derivative

$$D(\eta) \equiv \frac{d}{d\eta}\gamma(\eta) \tag{4.2}$$

where

$$\gamma(\eta) \equiv \frac{\mathcal{V}}{32\pi}\int_{-\infty}^{\infty}\frac{dp^{o}}{2\pi}\frac{e^{-ip^{o}\cdot\eta}}{|p^{o}|} \tag{4.3}$$

where $1/|p^{o}|$ must be understood as its Hadamard's finite part distribution (Campos and Verdaguer, 1996). Then the fluctuation-dissipation relation takes the form

$$N(\eta) = \int_{-\infty}^{\infty} d\eta'\, K_{FD}(\eta - \eta')\gamma(\eta') \tag{4.4}$$

where the fluctuation-dissipation kernel $K_{FD}(\eta - \eta')$ is given by the distribution

$$K_{FD}(\eta) \equiv \int_{-\infty}^{\infty}\frac{dp^{o}}{2\pi}\, e^{-ip^{o}\cdot\eta}|p^{o}| \tag{4.5}$$

as can be easily checked. This relation connects the effect of the quantum fluctuations of the "environment," which are represented by the noise kernel N, with the dissipation on the scale factor $a(\eta)$ as a consequence of the quantum particle creation (Calzetta and Hu, 1994).

4.2. Improved Effective Action

Let us now turn to the evaluation of the back-reaction equations, that is, the dynamical equations for the scalar field $a(\eta)$. In principle these equations can be derived from the CTP effective action using (3.3), i.e., $\delta\Gamma_{\rm CTP}/\delta a^{+}|_{a_\pm = a} = 0$, for the fields $a^{\pm}(\eta)$ (which are classical fields), using the $\Gamma_{\rm CTP}$ of (3.11) derived in the previous section. A problem might arise from the fact that this effective action has an imaginary part, since $S^{\rm R}_{\rm IF}$ is complex; see (3.13). It should be clear, however, that the imaginary part of $S^{\rm R}_{\rm IF}$ will not contribute to the field equation derived in that form because $\mathrm{Im}(S^{\rm R}_{\rm IF}[a^{\pm}])$ is quadratic in the difference $\Delta V(a) = V_{+}(a^{+}) - V_{-}(a^{-})$.

However, as we pointed out in the previous section, from the point of view of the system–environment relation the imaginary part of the influence action is related to the noise suffered by the system from the environment fluctuations. Thus we can improve the semiclassical back-reaction equations by taking into account such fluctuations. This may be formally achieved if we define the influence functional (Feynman and Vernon, 1963)

$$\mathcal{F}_{\rm IF}[a^{\pm}] = e^{iS^{\rm R}_{\rm IF}[a^{\pm}]} \tag{4.6}$$

and note that it may be written as

$$\mathcal{F}_{\rm IF}[a^{\pm}] = \int \mathcal{D}\xi\, \mathcal{P}[\xi] \exp\left\{ i\left[\mathrm{Re}(S^{\rm R}_{\rm IF}[a^{\pm}]) + \int d\eta\, \xi(\eta)\Delta V(\eta) \right]\right\} \tag{4.7}$$

where

$$\mathcal{P}[\xi] = \frac{\exp\left[-\frac{1}{2}\int d\eta\, d\eta'\, \xi(\eta)(\mathrm{N}(\eta - \eta'))^{-1}\xi(\eta') \right]}{\int \mathcal{D}\xi \exp\left[-\frac{1}{2}\int d\eta\, d\eta'\, \xi(\eta)(\mathrm{N}(\eta - \eta'))^{-1}\xi(\eta') \right]} \tag{4.8}$$

and we have used only a simple path-integral Gaussian identity. That is, performing the path integral in (4.7) with $\mathcal{P}[\xi]$ defined in (4.8) leads directly to (4.6). If we interpret $\mathcal{P}[\xi]$ as a Gaussian probability distribution for the field $\xi(\eta)$, where the kernel $\mathrm{N}(\eta - \eta')$ plays the role of noise, the action in (4.7) may be seen formally as the action for a field $a(\eta)$ which is coupled to an external stochastic source $\xi(\eta)$ (Feynman and Vernon, 1963; Feynman and Hibbs, 1965).

The influence functional (4.6) can be seen as the mean value with respect to the stochastic field $\xi(\eta)$ of an influence functional for an improved effective action $S_{\rm eff}$ defined by

$$S_{\rm eff}[a^{\pm}; \xi] = S^{\rm R}_{g,m}[a^{+}] - S^{\rm R}_{g,m}[a^{-}] + \mathrm{Re}(S^{\rm R}_{\rm IF}[a^{\pm}]) + \int d\eta\; \xi(\eta)\Delta V(\eta) \quad (4.9)$$

This comes from (4.7) and the addition of the gravitational and classical matter terms (3.12). Now the field $\xi(\eta)$ will act as a stochastic source in the improved semiclassical equation when the functional derivation with respect to $a^{+}(\eta)$ is taken. This stochastic field is not dynamical; it is completely defined by the following relations, which may be derived from the characteristic functional, i.e., the functional Fourier transform of $\mathcal{P}[\xi]$:

$$\langle \xi(\eta) \rangle_{\xi} = 0$$

$$\langle \xi(\eta)\xi(\eta') \rangle_{\xi} = \mathrm{N}(\eta - \eta') \quad (4.10)$$

Since the probability distribution is Gaussian, the noise kernel is the two-point correlation function of the stochastic field. In our case, as one can see from (3.10), the noise is white.

Before proceeding to the derivation of the improved semiclassical equations it is convenient for comparison with previous work (Hartle, 1981; Anderson, 1983, 1984) to write the effective action (4.9) in terms of dimensionless quantities such as χ, $b(\chi)$, and $\zeta(\chi)$ for the conformal time η, cosmological scale factor $a(\eta)$, and stochastic source $\xi(\eta)$, respectively. These are defined by

$$\chi = \frac{\tilde{\rho}_r^{1/4}}{6^{1/2}}\,\eta, \qquad b = \frac{a}{l_p\tilde{\rho}_r^{1/4}}, \qquad \zeta = \frac{\tilde{\rho}_r^{1/4}}{6^{1/2}}\,\xi \quad (4.11)$$

We also introduce dimensionless frequencies ω, renormalization parameter $\tilde{\mu}$, and volume $\hat{\mathcal{V}}$ instead of p^{o}, $\overline{\mu}$, $\mathcal{V}$, respectively, as

$$\omega = \frac{6^{1/2}}{\tilde{\rho}_r^{1/4}}\,p^{o}, \qquad \tilde{\mu} = 6^{1/2} l_p \overline{\mu}, \qquad \hat{\mathcal{V}} = \frac{\tilde{\rho}_r^{3/4}}{6^{1/2}}\,\mathcal{V} \quad (4.12)$$

Thus the improved effective action (4.9) for the dimensionless scale field $b(\chi)$ is

$$\begin{aligned} S_{\rm eff}[b^{\pm}; \zeta] \equiv \mathcal{V}6^{1/2}\tilde{\rho}_r^{3/4} \int d\chi\, \Big\{ & -\Delta(\dot{b}^2 + \hat{\xi} b)(\chi) \\ & + \frac{\nu^2}{32\pi^2}\left[\Delta\left(\ln b\left(\frac{\ddot{b}}{b}\right)^2\right)(\chi) - \Delta\left(\frac{\ddot{b}}{b}\right)(\chi)\kappa\left[\chi; \left\{\frac{\ddot{b}}{b}\right\}\right]\right] \\ & + \left(\frac{\nu}{\hat{\mathcal{V}}}\right)\zeta(\chi)\Delta\left(\frac{\ddot{b}}{b}\right)(\chi)\Big\} \end{aligned} \quad (4.13)$$

The nonlocal operator κ is defined by its action on a general function $f(\chi)$ by

$$\kappa[\chi; f(\chi)] \equiv \int d\chi'\, \mathrm{h}(\chi - \chi'; \tilde{\mu}) f(\chi') \tag{4.14}$$

where the kernel H in (3.14) is now reduced to

$$\mathrm{h}(\chi - \chi'; \tilde{\mu}) \equiv \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega\cdot(\chi - \chi')} \left[\ln \frac{|\omega|}{\tilde{\mu}} + \frac{i\pi}{2} \operatorname{sgn}(-\omega) \right] \tag{4.15}$$

i.e., it is the Fourier transform of $[\ln(|\omega|/\tilde{\mu}) + (i\pi/2)\operatorname{sgn}(-\omega)]$. On the other hand, the stochastic source $\zeta(\chi)$ is defined from (4.10) by

$$\langle \zeta(\chi)\zeta(\chi')\rangle_\zeta = \frac{\hat{\mathcal{V}}}{192\pi} \delta(\chi - \chi') \tag{4.16}$$

5. BACK-REACTION EQUATIONS

We are now ready to derive the dynamical equations for the cosmological scale factor in its dimensionless form $b(\chi)$ by functional derivation of the effective action (4.13) as

$$\frac{\delta}{\delta b^+} (S_{\text{eff}}[b^\pm; \xi]) \Big|_{b_\pm = b} \equiv 0 \tag{5.1}$$

After functional derivation and the identification of the plus and minus fields, the equation acquires an interesting form if we multiply it by $\dot{b}$. The final form is

$$\frac{d}{d\chi} \left\{ \dot{b}^2 - \hat{\xi} b - \frac{\nu^2}{32\pi^2} \ln b \left(\frac{\ddot{b}}{b}\right)^2 + \dot{b} \frac{d}{d\chi} \left[\frac{\nu^2}{16\pi^2} \left[\ln b \left(\frac{\ddot{b}}{b^2}\right) - \frac{1}{b} \kappa\left[\chi; \frac{\ddot{b}}{b}\right] \right] \right. \right.$$
$$\left. \left. + \left(\frac{\nu}{\hat{\mathcal{V}}}\right) \frac{\zeta(\chi)}{b} \right] \right\} = \left(\frac{\dot{b}}{b}\right) \frac{d}{d\chi} \left[-\frac{\nu^2}{16\pi^2} \kappa\left[\chi; \frac{\ddot{b}}{b}\right] + \left(\frac{\nu}{\hat{\mathcal{V}}}\right) \zeta(\chi) \right] \tag{5.2}$$

A few comments are now in order. First, the equation is a stochastic equation, given that the field $\zeta(\chi)$ is stochastic, but if we take its mean value with respect to that field ζ, the equation reduces to the more familiar semiclassical equation without the stochastic source. This stochastic semiclassical equation describes the dynamics of the cosmological scale factor when it is driven by a classical source of radiation and dust fluids [this is the origin of the first two terms in (5.2)], and it is coupled to a massless nonconformal field. The effects of the quantum field are those proportional to the coupling parameter ν. The terms with $\ln b$ come from the renormalization of the stress-energy momentum of the quantum field [see (2.7)] and then there are nonlocal contributions represented by $\kappa[\chi; \ddot{b}/b]$. The effect due to the production of quantum particles, in particular the dissipation of the field $b(\chi)$ due to such particle creation, is included in these nonlocal terms. The fluctuation-dissipation relation derived in (4.4) suggests, in fact, that these dissipative effects are to be found in the terms involving the kernel $D(\eta - \eta')$, which is only a part of the kernel $H(\eta - \eta'; \bar{\mu})$ in (3.14), or, equivalently, of $h(\chi - \chi'; \tilde{\mu})$ in (4.15). Nonlocal terms do not appear when there is no particle production; see, for instance, (Anderson, 1983, 1984, 1985, 1986).

Equation (5.2) is real and causal; this is clearly seen if we explicitly compute the Fourier transform in the definition (4.15) of the kernel $h(\chi - \chi'; \tilde{\mu})$:

$$h(\chi - \chi'; \tilde{\mu}) = -\left[Pf \frac{\theta(\chi - \chi')}{\chi - \chi'} + (\gamma + \ln \tilde{\mu})\delta(\chi - \chi') \right] \tag{5.3}$$

where γ is the Euler constant and Pf means the Hadamard finite part. It is clear that due to the $\theta(\chi - \chi')$ term in (5.3), when this is substituted into the nonlocal operator $\kappa[\chi; f(\chi)]$, the integration in (4.14) is restricted to $\chi' < \chi$; thus (5.2) is manifestly real and causal. Note that this is in fact the main difference between the CTP approach and Hartle's approach. Compare, for instance, (4.14) with the corresponding nonlocal operator (2.23) in Hartle (1981), which includes integration on all values of χ', i.e., to the future of χ also. The equation has derivatives higher than two in the terms of quantum origin. This is a well-known fact in the semiclassical context and it is rather common in back-reaction problems. The high derivative terms are responsible for spurious solutions; to eliminate such solutions, reduction of order methods have been suggested (Simon, 1991; Parker and Simon, 1993; Flanagan and Wald, 1996). The methods are neat when one deals with local equations, but when nonlocal equations are involved, as in our case, special care must be taken since there might be some ambiguity if integration by parts is made.

When quantum effects are ignored, i.e., when we take $\nu = 0$ in the back-reaction equation, it becomes $2\ddot{b} - \hat{\xi} = 0$, which is the classical equation when only the two classical fluids are present and admits the classical solution

$$b = \chi + \frac{1}{4} \hat{\xi}\chi^2 \tag{5.4}$$

with appropriate initial conditions. When baryonic matter is not present, i.e., $\hat{\xi} = 0$, the classical solution is $b = \chi$, and it is interesting to note that (5.2), when it is averaged with respect to $\zeta(\chi)$ also admits $b = \chi$ as a solution, which is in agreement with Hartle's solution. As remarked in the introduction, this is expected, as in such a case no particle creation is produced. Had we introduced local terms in the action of the type $R_{\mu\nu}R^{\mu\nu}$ (these are related to the previously described parameter ambiguity), this would not necessarily be the case.

The analytic solution of the stochastic integrodifferential equation (5.2) is not possible, but since $\hat{\xi}$, the parameter that gives the baryon-to-photon ratio, is assumed to be very small, it makes sense to find solutions linearized around a radiative universe. Thus we follow Hartle (1981) and look for solutions of the type

$$b(\chi) = \chi + \hat{\xi} g(\chi) \tag{5.5}$$

Substituting this into (5.2), taking only terms linear in $\hat{\xi}$, the equation can be integrated twice and becomes

$$g + \frac{\nu^2}{32\pi^2\chi}\left(\ln \chi \frac{\ddot{g}}{\chi} - \kappa\left[\chi; \frac{\ddot{g}}{\chi}\right]\right) + \frac{\nu}{2\hat{V}\hat{\xi}}\left(\frac{\zeta}{\chi}\right) = \frac{\chi^2}{4} + A_o\chi + B_o \tag{5.6}$$

where A_o and B_o are integration constants which may be taken to vanish. The term $\chi^2/4$ gives the expansion corresponding to a matter universe and comes from the second term in (5.2). The terms with ν are of quantum origin. Equation (5.6) is the dynamical equation for the perturbation $g(\chi)$ of the scale factor around a radiative classical solution.

This equation can now be directly compared with (3.9) in Hartle (1981). In our case we have the external stochastic source ζ that accounts for the fluctuations of the quantum stress-energy tensor, which was not considered previously, but if we take the mean value with respect of ζ, the resulting equation should be comparable with Hartle's equation. The main difference here, as we pointed out earlier, is that the equation is real and causal. But the basic structure of the equation is similar and the conclusions that Hartle draws in his analysis are essentially the same here.

Thus, ignoring the stochastic term $\zeta(\chi)$, we can conclude the following. For large-χ behavior, $g \sim \chi^2/4 + O(\ln \chi/\chi^2)$ is a self-consistent solution of equation (5.6). For small values of χ one finds that a solution $g \sim g_1\chi + O(\chi^{3+\epsilon})$, where $\epsilon > 0$ is also consistent with the equation. This guaranties the convergence of the integral in $\kappa[\chi; \ddot{g}/\chi]$ and is in agreement with the assumption that near the classical singularity the perturbation is small. This is imposed by requiring that $g'(0) = 0$, which is consistent with the assumption that the back reaction is only due to the creation of particles. In fact, in this case the universe starts classical, then creates quantum particles, and it is the back reaction of these quantum effects that we are considering.

The inclusion of the stochastic source in the equation is new and represents an improvement over the previous semiclassical analysis. From the fluctuation-dissipation relation one expects now that there should be an equilibrium between the dissipative effects due to particle creation and the stochastic source due to the quantum fluctuation of the environment in a way similar to what happens in the Brownian motion. The detailed study of the resulting stochastic behavior of the conformal scale factor described by (5.6) is made by Calzetta *et al.* (1997).

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